Gravitational Waves and Gravitons

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1 Classical Electromagnetism and Electromagnetic Waves

We begin this document with the classical picture of the electromagnetic (EM) field. We will show that the EM field satisfies a wave equation. We then go on to show that the EM field may be thought of a collection of harmonic oscillators at every point in space by examining the Hamiltonian of the field in Fourier space.

1.1 The Field and Wave Equations

In 1864, James Clerk Maxwell proposed that solutions of the wave-equations of classical EM could be identified with light waves. Mathematically, the vacuum form of the EM field equations are

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \times \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \end{aligned}$$

where we have used non-rationalized units to free the equations of constants. It is well known that the application of vector calculus to the field equations results in the two well known wave equations of classical EM.

$$\Box \mathbf{E} = 0$$

$$\Box \mathbf{B} = 0$$

where $\Box$ is the D'Alembertian defined as

$$\Box = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The physical interpretation of these wave equations is well known - disturbances in the EM field propagate through space in the form of waves of speed $c = 1$ (recall that in rationalized units, velocities are unit-less and the speed of light is unity).

1.2 The Hamiltonian of the EM Field

The scalar and vector potentials of EM are defined by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

We pick the transverse gauge

$$\nabla \cdot \mathbf{A} = 0$$

and in terms of the potentials, the wave equation becomes

$$\Box \mathbf{A} = 0$$

NB: In the transverse gauge, ϕ is completely determined byρ, the charge density, and has no differential equation in t - it is not a dynamically independent field. The independent modes of $\mathbf{A}$ are

$$\mathbf{A} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathbf{Q} e^{\imath \mathbf{k}\cdot\mathbf{r}} d^3k$$

With $\mathbf{k} = \omega\widehat{\mathbf{n}}$, the gauge condition and wave equation for $\mathbf{A}$ become

$$\widehat{\mathbf{n}} \cdot \mathbf{Q} = 0$$

$$\frac{\partial^2 \mathbf{Q}}{\partial t^2} + \omega^2 \mathbf{Q} = \mathbf{0}$$

for every value of $\mathbf{k}$. Now this has solutions, for $\mathbf{A}$,

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{\alpha=1}^{2} \int [\widehat{\epsilon_\alpha}(\mathbf{k}) \mathbf{c}_\alpha(\mathbf{k}, t) e^{\imath \mathbf{k}\cdot\mathbf{r}} + \widehat{\epsilon_\alpha^\star}(\mathbf{k}) \mathbf{c}_\alpha^\star(\mathbf{k}, t) e^{-\imath \mathbf{k}\cdot\mathbf{r}}] d^3\mathbf{k}$$

where we've defined orthonormal and transverse unit vectors $\widehat{\epsilon_\alpha}(\mathbf{k})$ such that

$$\widehat{\epsilon_\alpha^\star}(\mathbf{k}) \cdot \widehat{\epsilon_\beta}(\mathbf{k}) = \delta_{\alpha\beta}$$

and

$$\widehat{\epsilon_\alpha}(\mathbf{k}) \cdot \widehat{\mathbf{n}} = 0$$

with $c_\alpha(\mathbf{k}, t)$ defined so that

$$\mathbf{c}(\mathbf{k}, t) = \sum_{\alpha=1}^{2} \widehat{\epsilon_\alpha}(\mathbf{k}, t) c_\alpha(\mathbf{k}, t)$$

Precisely why we've done all of this will become clear in the next section, but, continuing on, we now use this formalism to express the classical Hamiltonian of the electromagnetic field

$$H = \frac{1}{8\pi} \int (\mathbf{E}^2 + \mathbf{B}^2) d^3r$$

as

$$H = \sum_{\alpha=1}^{2} \int \mathcal{H}_\alpha(\mathbf{k}) d^3k$$

with

$$\mathcal{H}_\alpha(\mathbf{k}) = \frac{\omega^2}{2\pi} c_\alpha^\star(\mathbf{k}, t) c_\alpha(\mathbf{k}, t) = \frac{\omega^2}{2\pi} \mid c_\alpha(\mathbf{k}) \mid^2$$

If we now define canonical coordinates

$$x_\alpha(\mathbf{k}, t) = \sqrt{\frac{1}{4\pi}} (c_\alpha(\mathbf{k}, t) + c_\alpha^\star(\mathbf{k}, t))$$

and

$$p_\alpha(\mathbf{k}, t) = -\imath \sqrt{\frac{\omega^2}{4\pi}} (c_\alpha(\mathbf{k}, t) - c_\alpha^\star(\mathbf{k}, t))$$

we can rewrite our Hamiltonian as

$$H = \sum_{\alpha=1}^{2} \int \mathcal{H}_\alpha(\mathbf{k}) d^3k$$

with

$$\mathcal{H}_\alpha(\mathbf{k}) = \frac{1}{2} [p_\alpha(\mathbf{k}, t)^2 + \omega^2 x_\alpha(\mathbf{k}, t)^2]$$

with Hamilton's Equations of Motions

$$\frac{dx_\alpha(\mathbf{k}, t)}{dt} = p_\alpha(\mathbf{k}, t)$$

and

$$\frac{dp_\alpha(\mathbf{k}, t)}{dt} = -\omega^2 x_\alpha(\mathbf{k}, t)$$

Hence the EM field may be represented by a collection of oscillators at every location in space, with a simple harmonic oscillator for every mode of oscillation. This is an extremely important result for it allows us to guess how things change when we quantize the field.

2 Quantized EM and Photons

Before we begin the arduous task of quantizing the field, it is natural to ask why it is desirable to do so. The answer is twofold - quantizing the field allows us to correctly calculate EM interactions between particles and it also offers the possibility of answering questions about the accelerating expansion of the universe!

2.1 Canonical Quantization

We begin by transforming the canonical coordinates used in the Hamiltonian of the previous section into operators. They must then satisfy the usual commutation relations

$$[x_\alpha(\mathbf{k},t), x_{\alpha'}(\mathbf{k}',t)] = [p_\alpha(\mathbf{k},t), p_{\alpha'}(\mathbf{k}',t)] = 0$$

and

$$[x_\alpha(\mathbf{k},t), p_{\alpha'}(\mathbf{k}',t)] = \imath\delta_{\alpha\alpha'}\delta_3(\mathbf{k}'-\mathbf{k})$$

We then define the raising and lowering operators by

$$a_\alpha = \sqrt{\frac{1}{2\omega}}[\omega x_\alpha + \imath p_\alpha]$$

and

$$a_\alpha^\dagger = \sqrt{\frac{1}{2\omega}}[\omega x_\alpha - \imath p_\alpha]$$

The raising and lowering operators satisfy the usual commutation relations

$$[a_\alpha(\mathbf{k},t), a_{\alpha'}(\mathbf{k}',t)] = [a_\alpha^\dagger(\mathbf{k},t), a_{\alpha'}^\dagger(\mathbf{k}',t)] = 0$$

and

$$[a_\alpha(\mathbf{k},t), a_{\alpha'}^\dagger(\mathbf{k}',t)] = \imath\delta_{\alpha\alpha'}\delta_3(\mathbf{k}'-\mathbf{k})$$

The Hamiltonian then becomes

$$H = \int \sum_{\alpha=1}^{2} \mathcal{H}_\alpha(\mathbf{k})$$

with

$$\mathcal{H}_\alpha(\mathbf{k}) = \frac{1}{2}\omega[a_{alpha}(\mathbf{k},t)a_{alpha}^\dagger(\mathbf{k},t) + a_{alpha}^\dagger(\mathbf{k},t)a_{alpha}(\mathbf{k},t)]$$

If we define the number operator by

$$\mathbf{N}_\alpha(\mathbf{k}) = a_\alpha^\dagger(\mathbf{k})a_\alpha(\mathbf{k})$$

then we may rewrite the Hamiltonian for each mode as

$$\mathcal{H}_\alpha(\mathbf{k}) = \omega\left(\mathbf{N}_\alpha(\mathbf{k}) + \frac{1}{2}\delta_3(0)\right)$$

If we label the eigenstates of this Hamiltonian by $n_\alpha(\mathbf{k})$ then each state has energy

$$E = \sum_{\alpha=1}^{2} \int \omega\left(n_\alpha(\mathbf{k}) + \frac{1}{2}\delta_3(0)\right) d^3k$$

2.2 Photons

We may identify the state $n_\alpha(\mathbf{k})$ with the occupation number of the mode $(\alpha, \mathbf{k})$. An increment of $n_\alpha(\mathbf{k})$ by 1 increases the energy by ω, so the eigenstates are $n_\alpha(\mathbf{k})$ photons with momentum $\mathbf{k}$. The creation and annihilation operators now add or remove photons from our quantized field. What about the ground-state energy? We have $n_\alpha(\mathbf{k})$ photons each with a minimum non-zero energy. The total energy thus becomes

$$E = \sum_{\alpha=1}^{2} \int \frac{\omega}{2} \delta_3(0) d^3k$$

This is troublesome! We seem to have a non-zero energy in the vacuum (no photons) state. It turns out that this is merely the first of many infinities that crop up in quantum field theory. The normalization of the relativistic wave function is another example of a troublesome infinity. Fortunately, this non-zero vacuum energy is easier to dispose of. The usual method is to reduce the energy of each mode so that the ground state has zero energy. The energy of our field in vacuum with no photons now sums to zero. Another argument is to identify the vacuum energy with the cosmological constant that appears in Einstein's Field Equation for Gravity (G). We will postpone a discussion of the cosmological constant to the section on General Relativity. Before ending, we shall point out (without proof), that photons are spin-1 bosons because they arise from the quantization of a vector field i.e. a $(1, 0)$-tensor field. In general, the quantization of a $(n, 0)$-tensor field results in the quanta of the field having spin-n.

3 Classical Gravitation and Gravitational Waves

We will now examine the celebrated Gravitational (G) field equations due to Einstein in 1916. Before we may do so, we shall present a quick review of the mathematical language of general relativity. It is assumed that the reader is familiar with the Einstein summation convention and with basic tensor calculus.

3.1 The Mathematics of Curvature

Given an n-dimensional manifold, $\mathcal{M}$, one defines the infinitesimal distance element to be

$$ds^{(2)} = g_{\mu\nu} dx^\mu dx^\nu$$

where $g_{\mu\nu}$ is the metric. Partial derivatives are replaced by covariant derivatives defined by

$$\nabla_\mu = \partial_\mu + \Gamma^\nu_{\mu\lambda}$$

where $\Gamma^\nu_{\mu\lambda}$ are the Christoffel symbols. They are related to the metric by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$$

In a sense, the Christoffel symbol corrects the partial derivative for the presence of curvature to give the correct covariant derivative. Straight lines on the manifold are geodesics and satisfy the geodesic equation.

$$v^\mu \nabla_\mu v^\nu = 0$$

The curvature of the manifold is quantified by the Riemann curvature tensor given by

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

The Riemann curvature tensor measures the change in a vector when parallel transported along two geodesics. The symmetries of the tensor results in the number of independent components being given by

$$N = \frac{n^2(n^2-1)}{12}$$

which, in the 4-dimensional spacetime that we live in, gives 20 independent components. The Riemann tensor may be contracted to obtain the Ricci tensor

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$$

The Ricci tensor measures distortions of the volume element due to the curvature of the manifold. The Ricci tensor is symmetric, implying that it has 10 free components in 4-dimensional spacetime. A further contraction of the Ricci tensor yields the scalar curvature

$$R = R^\mu_\mu$$

Finally, from the Ricci tensor and the scalar curvature, one may construct the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

Since the Einstein tensor is constructed from the Ricci tensor, it too measures the distortions of volumes on the manifold and has 10 free components in 4-dimensional spacetime. With this, we end our survey of the mathematics of curvature and delve into the G field equations.

3.2 The Field and Wave Equations

The General Theory of Relativity put forward by Albert Einstein in 1916 is a three part replacement for Newtonian mechanics and Gravity. First, the Galilean transforms are replaced by Lorentz transforms

$$x^{\mu'} = \Lambda^{\mu'}_\nu x^\nu$$

where the $\Lambda^{\mu'}_\nu$ are the Lorentz matrices belonging to the Poincaré group of boosts, rotations and translations. Next, Newton's First Law is generalized to state that objects travel along geodesics when no force is acting on them. Geodesics are given by

$$v^\mu \nabla_\mu v^\nu = 0$$

Lastly, Newton's Law of Universal Gravitation is replaced by the Einstein Field Equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu} - \Upsilon g_{\mu\nu}$$

where Υ is the Cosmological Constant and $T_{\mu\nu}$ is the energy-momentum tensor. The *modus operandi* is thus to use the action principle to derive an energy-momentum tensor for the physical situation being modeled and to use it to calculate the Einstein tensor. From the Einstein tensor, one may obtain the Christoffel connections and hence the metric. At this point, the metric can be used along with the geodesic equation to calculate the trajectories of free particles. Thus, in the general theory, gravity is not an independent field defined on spacetime as much as it is the curvature of spacetime itself. This will prove to be the primary stumbling block in the quantization of gravity. Going back to the Cosmological Constant Υ, it is interesting to recall that Einstein first introduced the constant to allow for steady state solutions of the Universe (i.e. neither expanding nor contracting) for at the time it was not known that the Universe was expanding (Hubble's seminal result demonstrating the expansion of the Universe came much later in 1929). Einstein removed the constant from his equations upon hearing of Hubble's result, calling the introduction of the constant his 'greatest blunder'. However, recent results from the Wilkinson Microwave Anisotropy Probe (WMAP) and Sloan Digital Sky Survey (SDSS) suggest that the expansion rate of the Universe is actually increasing leading to the reintroduction of the Cosmological Constant in the field equations. It is suggested that part of the contribution to the measured value of Υ comes from the non-zero vacuum energy of the quantized EM field, coupling the EM field to the shape of the Universe. Returning to the field equations, in the absence of a source for the G field, and ignoring the contribution of Υ for the time being, one finds that the vacuum G field equation is

$$G_{\mu\nu} = 0$$

Recall that $G_{\mu\nu}$ has ten free components. Since we may pick our coordinate axii arbitrarily, we can further reduce (the details are messy) the number of independent components to a mere 6 second order hyperbolic-elliptical PDEs. Several specific solutions for various energy-momentum tensors are known, but no general solution exists. To search for wavelike solutions, it is common practice to look at the weak field, low velocity limit of the field equations. The first step involves breaking the general spacetime metric into a 'flat' Minkowski metric along with a small perturbation.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where

$$\mid h_{\mu\nu} \mid \leq 1$$

The *traceless strain tensor* is then defined to be

$$s_{ij} = h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij}$$

Note that the indices i and j appearing in the strain equation run over only spatial components i.e. $i, j \in \{1, 2, 3\}$. The strain tensor therefore measures the spatial distortions of the perturbations of the metric. The time-like part of the perturbation metric (i.e. the 00-component) is a scalar. The weak-field versions of the field equations are

$$G_{00} = 8\pi T_{00}$$
$$G_{0j} = 8\pi T_{0j}$$
$$G_{ij} = 8\pi T_{ij}$$

Notice how this version of the field equations decouple the time and space components of the field equation. Applying this version of the field equations to the strain tensor, one can show that the strain tensor satisfies the wave equation

$$\Box s_{ij} = 0$$

This is the wave equation for G radiation. *The G field equations therefore predict the existence of G waves, just as the EM field equations predict the existence of EM waves.*

3.3 G Waves

Without going into mathematical details (for the math gets tedious), we may appreciate some salient features of G waves. First, one may ask, what the effect of G wave on a massive substance would be. That G waves should interact with massive substances may be inferred from the fact that G waves are distortions of the metric which is in turn also distorted by the massive object that the G wave interacts with. This is analogous to the interactions of an EM wave with charged objects such as an electron. The effects of a G wave may best be visualized by looking at the case of a G wave passing perpendicularly through the plane of a massive ring. The effect of the G wave is to alternately stretch and squeeze the ring of mass along two perpendicular axii. It turns out that just as we have two linearly independent polarizations of EM waves at an angle of $\frac{\pi}{2}$ to each other, we also have two linearly independent polarizations of G waves. However, these occur at an angle of $\frac{\pi}{4}$ to each other. This is because the underlying field that describes these waves is a tensor field and hence the polarization exhibited by G waves is a sort of 'tensor polarization' that points in two directions. The two linearly independent polarizations are referred to as the $+$ polarization and the $\times$ polarization. Elliptically polarized G waves that are time dependent linear combinations of these two polarizations are therefore also possible. G waves are described by their amplitude, frequency (ν), and wavelength (λ). The amplitude of a G wave is a measure of the fractional stretching (or squeezing) that it produces in a body. The wavelength and frequency of a G wave are defined in the usual way and satisfy $\lambda\nu = 1$ since they travel at the speed of light. Like EM waves, G waves carry energy away from the source of the wave and deposit part of this energy when they interact with a massive body.

3.4 Experiments

Experimental confirmation of G waves was first provided by Russell Hulse and John Taylor for which they were awarded the 1993 Nobel Prize in Physics. They examined the rotation rate of the system **PSR B1913 + 16** which consists of a gravitationally bound star and pulsar (rotating neutron star with huge amounts of angular momentum). Due to the very high density of neutron stars and tiny radius of orbit, the G field of the system is very strong and as a result the system is a strong emitter of G radiation. This carries energy away from the system causing the radius of the orbit to decrease, reducing the time period of the system within measurable time scales. Although this proves the existence of G waves, G waves have not as yet been detected directly. The reason lies in the fact that G waves carry very small amounts of energy and have minuscule due to the weakness of gravity itself. For example, calculations indicate that the Earth emits a minuscule $300W$ of gravitational radiation as it travels around the Sun. The most massive binaries - black hole binaries are expected to radiate considerably larger amounts of G radiation and it is expected that events involving the merger of two black holes should emit G radiation with a large enough amplitude to be detected by the latest generation of G wave detectors such as LIGO. Traditional detectors take the form of a Weber bar surrounded by piezoelectric crystals. A passing G waves would distort the bar more than it would distort the piezoelectric crystals causing measurable voltages to be set up across the crystals. Such experiments have proven unsuccessful so far and the latest generation of G wave detectors are based on the principle of interferometry (though one Weber sphere type experiment, MiniGrail, is in progress in the Netherlands). In interferometric detectors such as LIGO, light is sent down two long, perpendicular tunnels multiple times before being allowed to interfere. G waves passing perpendicular to the plane of the arms would alternately squeeze and stretch the arms as well as the light beam itself. However, since the stretching and squeezing is proportional to the length of the body, the arms would change length by a much greater factor than the change in the wavelength of the light beam causing an interference pattern to appear. Several such experiments are either already operational or in the process of reaching operational status. The presence of multiple detectors suggests that it may become possible to triangulate G wave sources allowing for correlations between G wave, neutrino and EM wave measurements. Eventually, it is hoped that the placement of interferometric detectors in Solar orbit, such as the LISA proposal, will allow for G wave astronomy. The future looks bright in G radiation!

4 Quantized Gravitation and Gravitons

We conclude this paper with a brief overview of quantum gravity and gravitons. So far, all attempts at a quantum theory of gravity have been failures. Einstein himself was the first to suggest that his field equations required quantum corrections in 1916 at the birth of the general theory. in 1927 Oskar Klein suggested that a quantum theory of gravity had to modify the nature of spacetime itself. The first technical papers on QG appeared in the works of Léon Rosenfeld in the early thirties. The graviton as the quantum of G radiation first appeared in a 1934 paper by Dmitrii Ivanovich Blokhintsev and FM Gal'perin which also showed that just as the photon is a spin one particle because the underlying field is a vector field, the graviton should be a massless a spin two particle because the underlying field is the metric tensor. Research into a quantum theory of gravity has been the focus of many famous theoreticians since and today the foremost candidates for a quantum theory of gravity are string theory and variants such as brane and M-theory and loop quantum gravity.

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